Cantor and the Infinite Stairway

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The infinite has perplexed philosophers for millennia. While it is difficult to fully gauge the feelings of ancient humanity on the matter, it is safe to speculate the Paleolithic and early Neolithic humanity probably thought of numerical quantity without any notion of a dualism between the finite and the infinite. The latter concept, if it had any meaning, would manifest in the world of the immediate.[1] The same could be said of later philosophies deemed by historians to be "Pantheistic." It was with the arising of complex mathematical systems and systems of counting that the question of whether quantities would magnify forever occurred to philosophers. The Greeks were skeptical on the matter, while Hindu mathematics posited very advanced mathematical views on infinity. Hebrew philosophy tended to reserve "the Infinite" to God alone, a trend that continued into Christian and Islamic thought. It was with the Renaissance that interest in mathematical concepts of infinity undertook a rebirth with the logical proofs of Galileo that set the stage for later developments in Set Theory. Finally, the infinite became a fully differentiated entity, possessing levels of value, with the birth of modern philosophical mathematics divorced from quantitative mathematics that happened in the late nineteenth century. Georg Cantor, a mystic whose yearnings hearkened more to the Vedic philosophies of old than most of nineteenth century thinking, became the unlikely star of the unfolding of modern Set Theory and its rigorous constructions of infinity that happened as the twentieth century was just around the corner.

One wonders whether Cantor reached back into antiquity for any of his ideas. Amir Aczel's speculation that Cantor got many of his ideas from Kaballah is interesting if only as an ironic historical contrast to the rationalist bent of later Set Theory.[2] What can be said without controversy is that Cantor inherited his concept of infinity from a long evolution of historical thought in western civilization that began with the skeptical Greeks, evolved with the pragmatic rationalist Greco-Roman Stoics and Epicureans, entered as central to western religious debates with medieval scholastic Christianity, and saw its birth as a serious object of western mathematics with Galileo. Yes, it was Galileo who brought infinity into the mainstream of western mathematics, a little known contribution he made to human thought after his explorations in Astronomy were closed off by Holy Censorship.

Galileo gave birth to the idea of a proof by one-to-one correspondence. Galileo speculated that if two lengths divided by finite segments correspond in an isomorphic sense, then they would correspond if divided into an infinite number of segments. Utilizing this assumption, and the assumption of actually existing mathematical infinity, Galileo noted that one could pair the counting number series with its ordered subset in a way that establishes that they have equal measure at infinity. He demonstrated that the counting numbers pair exactly with the even numbers, even though it seems as if there should be more of the former than the latter:

1 2 3 4 5 6 7 8 9 10 11 …. Infinity

2 4 6 8 10 12 14 16 18 20 22…. Infinity

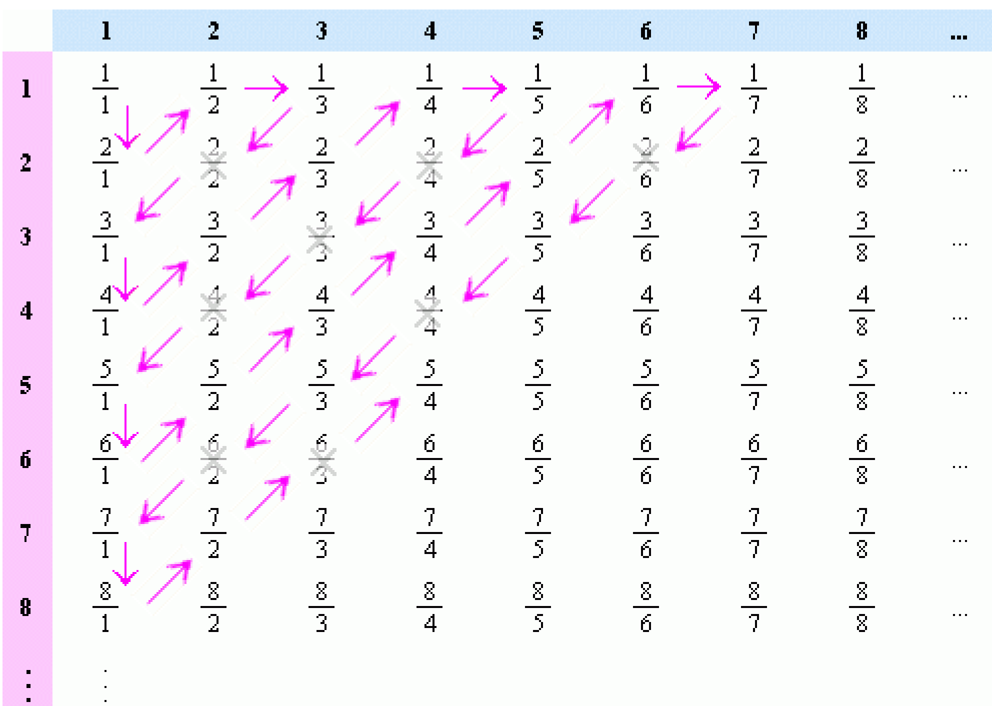
One will notice that the counting numbers "pair" with the even numbers as they both extend in to infinity. The seeming incongruity of the even numbers, being a lesser subset of the counting numbers in finite quantities, then being shown as exactly congruent measures at infinity was a paradox that boggled the mind. Yet, it followed from the logic of infinity as it occurred to Galileo.

Infinity entered the realm of mathematical speculation as a topic fully divorced from theology with the rise of rationalism and empiricism in the later modern period. Newtonian Calculus used proofs that relied very heavily on belief in the infinite and infinitesimal. Calculus was the basis of a physics that held to a belief in a continuous Universe that was not divided into discrete units, an assumption that would come to a crashing re-evaluation with quantum mechanics but which seemed evident in the world of Newton (as well as our daily experience). It was the philosopher Berkeley who presented a major challenge to those secular thinkers who challenged the existence of God as apparently resting on faith over experiment but then assumed the existence of mathematical infinity which also apparently rested on faith over scientific skepticism. Mathematicians responded to Berkeley's arguments by coming up with the notion of limits instead of an actual belief in infinity. The idea of limits was then incorporated into Calculus as much as a response to the philosophical arguments of Berkeley as any need for the concept mathematically. "As the **limit** of X goes to infinity, 1/X goes to zero" was preferred to "As X goes to infinity, 1/X goes to zero." A measure of skepticism about the existence of actual infinity was seen as the hallmark of a rational mind among empiricist philosophers.

Skepticism about metaphysical matters such as infinity lasted well in to the nineteenth century as the mechanistic world-view fully came into its own and the physics of Newtown eclipsed the organic spiritual world-view to which Newtown still clung.[3]

Ironically, it was as the nineteenth century was ending (with Newtonian physics just about to turn the quantum corner that would jettison continuity but revive other uses for infinity in its "renormalization" methods) that mathematics rediscovered infinity. Georg Cantor was as much a mystic as a mathematician. For Cantor, infinity was a vivid reality. In an intellectual climate of skepticism, he strongly believed that **The Absolute** was guiding him in his mathematical discoveries, just as Newton felt a connection with God, Ramanujan felt his connection to Hindu mysticism, and many quantum physicists have looked to Eastern thought. Cantor did not let the idea of "limits" get in the way of what he felt was an experiential approach to the infinite. He was willing to look infinity in the eye. It was believed in his lifetime that he was insane, a belief leading to several stays in mental institutions. Yet, Cantor's mathematical insights soon won the respect of his peers and formed the basis of later developments in mathematical Set Theory. Cantor was the first to discover the idea that there were levels of infinity, an idea that leads to a somewhat paradoxical view of the infinite but which for Cantor was a pattern of infinite beauty.

Cantor first extended Galileo's notion of paired infinities to the issue of rational numbers and counting numbers to show that the two pair.[4] It is here that the question of infinity is truly mysterious. Common sense would tell us that there definitely should be more rational numbers than counting numbers. Yet, Cantor was able to pair the rational with the counting numbers.



One can clearly see from the example above that the pattern would give all possible rational numbers that could ever exist if one extends it to infinity. If one looks at the eighth diagonal, one begins with 1/8, then 2/7, then 3/6, 4/5, 5/4, 6/3, 7/2, then 8/1. The reader will notice that as one then looks at all previous diagonals, 1 through 7, then including 8; one sees all possible fractions that could be constructed from numbers 1 through 8. The ninth diagonal would then give us all possible rational fractions using 1 through 9. (Some repeat) Extending this to infinity will give us all possible rational numbers that could ever exist. Yet, one notices that if we were to take each diagonal and stretch them out by following the arrows shown:

1/1 2/1   1/2 1/3   2/2 3/1   4/1 3/2 ….

One can pair these to the counting numbers! Since one can count the rational numbers with the counting numbers, this means that the rational numbers are “countably infinite” in the sense that they pair with the counting numbers.

Cantor earned the enmity of some, including the mathematician Leopold Kronecker who clung tenaciously to mathematical finitism. However, by and large, mathematicians accepted his findings and a revolution in mathematical philosophy ensued. Yet, it was the discovery of the idea of levels of infinity that really shook the world of philosophy. It was not long after Cantor discovered the counter-intuitive idea that there are not infinitely more rationals than counting numbers, Cantor then discovered a proof that the entire set of real numbers are **NOT** a countable infinity, that in fact there are more real numbers than rational numbers! I will state his proof below:

Let us assume that we can arrange all real numbers, irrational as well as rational, and pair them with the counting numbers. We can do so below by listing rows, and then putting real numbers in a random order.

Row:

1) **1.233456987644…..**
2) **2.547894533238……**
3) **8.438764399997…..**
4) **7.987435234563…..**
5) **3.876540984321…..**

And so on to an infinite number of rows….

Numbers in the millions would have to be arranged so that the decimals would line up. Now, please choose the one's place in

the first row, the tenth's in the second row, the hundredth's place, in the third row, continuing to infinity and we get a real number 1.5375…. which is presumably listed somewhere down the list of rows. Where it would be is of no consequence since we are assuming the possibility of pairing and we can leave it unstated if indeed our goal is to disprove our initial assumption. We will continue by adding one to each decimal number. 1 becomes 2, 5 becomes 6, 9 becomes 0 (a 10 of sorts) and we get a real number that is 2.6486… which is certainly found somewhere if all real numbers pair. But, alas, there is a problem. The first number of this new real number is one more than the number in the first row, the second number is one more than the second digit of the second number, the third digit is different than the third digit of the third row, until we see a series of differences extending to infinity. We have an entirely unique number that is not found anywhere!

We have contradicted our premise that we can pair all real numbers to the counting numbers. Our falsified assumption means that we have an infinity that cannot be counted. One could say, in essence, that there are more real numbers than rational numbers. We have infinities that are countable, and then other infinities that are uncountable.

Cantor eventually proved that the infinity of the Real Numbers was the "Power Set" of the infinity of the Counting Numbers. The proof relied on a binary code in which all real numbers could be coded with zeroes and ones. Since there is a countable infinite number of decimal places in any base system, including binary, then the number of real numbers corresponds to 2 to the power of the infinity that corresponds to counting numbers (2 to the power of the infinity of the counting numbers). This opened a question as to whether there were levels of infinity between the Rationals and the Reals. Cantor did not believe so but could never prove his belief in the idea that the Rationals and the Reals corresponded to two levels of infinity with no levels in between, known to history as his "Continuum Hypothesis." The goal of proving his quest eventually took a toll on his nerves and Cantor was institutionalized due to the stress of never achieving what he increasingly saw as a religious quest to pierce the nature of Infinity itself.

Philosophers and mathematicians were shaken to their core and Set Theory was fully prepared to take off in its twentieth century revolution that shifted mathematics from being essentially quantitative to being a philosophical exercise, and for some even a metaphysical endeavor. Less mystical and more rational forms of Set Theory emerged by philosophers such as Bertrand Russell who, while they respected the insights of Cantor, wanted a system shorn of Cantor's own belief in Absolute Infinity.[5] Set Theory was set down in very rigorous form and passed into the world of respectable mathematics once it could fully divorce mathematical infinities from metaphysical concepts of infinity, a divorce that would have saddened Cantor himself. Cantor's own speculation that there were no infinities between the Rationals and the Reals was proven to be undecidable by standard Set Theory assumptions. This was a blow to Cantor's own philosophy of Mathematical Platonism, the idea that mathematics is "real," given that no new assumptions were invoked to try to decide the issue one way or the other. For much of the twentieth century, Pure Mathematics moved closer to closer to abstract philosophy precisely as Computational Mathematics became the hand-maiden of the sciences. The older dream of a unified mathematical reality that inspired previous generations of mathematicians seemed to recede.

Modern philosophers and scientists have tended to be nominalist in approaching the question of whether mathematical infinities are "real" or not. Most incline to finitism, the belief that infinity has no reality to it and that the most we can discuss are limits and approximations. Yet, there is an interesting footnote to the whole question. Recently physicists have stopped trying to "renormalize" infinities out of existence and have actually accepted the possibility of an infinite multiverse. These cosmologists have suggested that we may live in an infinite Universe after a long period of believing in a Universe that is finite but unbounded along the "de Sitter" model. Their efforts have re-opened the philosophical debates around infinity. The possibility that infinity might actually have a place in our cosmological models begs the question of whether or not we can actually base our cosmological theories on probability arguments based on infinity.[6] Cantor and Set Theory may become relevant to the debates around cosmology, the multiverse, and our understanding of energy as would never have been considered possible during most of the twentieth century. The final word has not been said on the relevance of Cantor. It may well be that his metaphysical manuscripts need to be dusted off and treated with full respect. Yesterday's insanity may become today's cosmology. It would not be the first time as we look across the history of modern science beginning with Galileo and his brazen arguments.

On a personal note, I will dedicate this essay to Jason Zarri, our recently deceased former editor here at Scholardarity. I first met him when I gave a talk at my Astronomy Club on the mathematical contributions of Galileo to an audience that was surprised that Galileo was not just an Astronomer but a mathematician also. When I presented the concepts of Set Theory, Jason was a responsive member of the audience who knew the subject well. I was frankly surprised at his familiarity with philosophical ideas but my surprise soon gave way to great respect. It was the beginning of a short and yet very beautiful friendship. His death, while tragic, should not overshadow the inspiration of his life. Jason encouraged all of us to reach to infinity.

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[1] For early man, for example, God, the Infinite, infused the world of Nature directly, as we see in the earliest parts of Genesis where God walks in the Garden. In later Books and concepts of the Bible God seems more removed from Nature.

[2] Azcel, Amir. Mystery of the Aleph, Pocket Books, 2000.

[3] One wonders what the mystical Newtown would have thought of the "Newtonian" world-view of the nineteenth century. Perhaps he would have echoed Marx in exclaiming, "I am not a Newtownian!"

[4] It is unclear whether Cantor knew of Galileo's speculation on infinity.

[5] Cantor's Paradox holds that there could be no absolute Infinity since such an Infinity would be less than its own Power Set.

[6] Physicists are inclined to say "yes," while Mathematicians are skeptical.